Classical intermittency and the quantum Anderson transition

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We investigate the properties of quantum systems whose classical counterpart presents intermittency. It is shown, by using recent semiclassical techniques, that the quantum spectral correlations of such systems are expressed in terms of the eigenvalues of an anomalous diffusion operator. For certain values of the parameters leading to ballistic diffusion and $1/f$ noise the spectral properties of our model show similarities with those of a disordered system at the Anderson transition. In Hamiltonian systems, intermittency is closely related to the presence of cantori in the classical phase space. We suggest, based on this relation, that our findings may be relevant for the description of the spectral correlations of Hamiltonians with a classical phase space homogeneously filled by cantori. Finally we discuss the extension of our results to higher dimensions and their relation to Anderson models with long-range hopping.

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The quantum properties of a disordered system, namely, a noninteracting particle in a random potential are strongly affected by both the dimensionality of the space and the strength of disorder. In less than three dimensions the wave functions are localized in the thermodynamic limit for any amount of disorder. In three and higher dimensions there exists a metal insulator transition (MIT) for a critical amount of disorder. Thus for disorder below the critical one the wave functions are extended through the sample, the Hamiltonian is accurately approximated by a random matrix with the appropriate symmetry, and the spectral correlations are given by Wigner-Dyson (WD) statistics [1]. In the opposite limit, wave functions are exponentially localized and the spectral correlations are described by Poisson statistics. A similar situation occurs in "quantum chaos." The celebrated Bohigas-Giannoni-Schmit conjecture [2] states that the WD statistics applies to the spectral correlations of quantum systems whose classical counterpart is chaotic. On the other hand, it is broadly accepted [3] that Poisson statistics describes the spectral correlations of quantum systems whose classical counterpart is integrable.

Deviations from WD statistics due to wavefunction localization has been intensively investigated in recent years. In disordered systems, they are expressed through the dimensionless conductance $g=E_c/\Delta$ where $E_c=\hbar/t_c$ is the Thouless energy, t_c is the classical time to cross the sample diffusively and Δ is the quantum mean level spacing. In the metallic regime $g\to\infty$ and WD statistics applies. Nonperturbative corrections due to a finite $g\gg 1$ were recently evaluated by Andreev and Altshuler [4] in the framework of the supersymmetry method [5]. They managed to express the two level spectral correlation function in terms of the spectral determinant of the classical diffusion operator. In the context of quantum chaos deviations from WD statistics are expected due to the nonuniversality of short periodic orbits (here we do not discuss other sources of deviation as dynamical localization or a mixed phase space). In a recent development [6], the two level spectral correlation function encoding such deviations was found to be related to the spectral determinant of the classical Perron-Frobenius operator which controls the evolution of the classical phase space. Unfortunately explicit results are hard to obtain since there is no general recipe to compute the eigenvalues of this operator for a generic classical Hamiltonian.

As disorder strength further increases $g\sim 1$ localization effects become dominant and eventually the system undergoes a MIT. At the MIT, the wave function moments P_q present anomalous scaling with respect to the sample size [7] L, $P_q=\int d^d r|\psi(\mathbf{r})|^{2q}\propto L^{-D_q(q-1)}$, where D_q is a set of exponents describing the transition. Wave functions with such a nontrivial scaling are said to be multifractal (for a review see Ref. [8]). Spectral fluctuations at the MIT (commonly referred to as "critical statistics" [9]) are intermediate between WD and Poisson statistics. Typical features include: scale invariant spectrum [10], level repulsion, and sub-Poisson number variance [11]. Different generalized random matrix model have been successfully employed to describe critical statistics [12].

A natural question to ask is whether critical statistics is related to any kind of classical motion. We shall show that, for a certain range of parameters, the spectral correlations of quantum systems whose classical counterpart presents intermittency [13] and $1/f$ noise are described by critical statistics. This is the main result of this work. We also suggest that our results may be useful to describe the spectral correlations of non-KAM Hamiltonians with a classical phase space homogeneously filled by cantori. The organization of the paper is as follows: Classical intermittency is introduced by studying the evolution of a simple nonlinear map. In the context of Hamiltonian systems, we also discuss its relation with classical phase space structures Quantum spectral correlations associated with classical intermittency are then investigated by using the above mentioned semiclassical Andreev-Altshuler formalism. Finally we discuss the extension of our results to higher dimensions and its relation with Anderson models with long range hopping.

I. CLASSICAL INTERMITTENCY

The phenomenon of intermittency is characterized by long periods of laminar (regular) motion interrupted by short irregular bursts [13]. As a simple example of a dynamical system with such features we investigate the following map on the real line [13,14]: $x_{n+1}=f(x_n)$, where $f(x)$ verifies $f(x)=-f(-x)$ and $f(x+N)=f(x)+N$ with N an integer. With the above rules the map needs to be defined only in a restricted interval $f(x_n)=(1+\epsilon)x_n+a|x_n|^z-1$, $0<x_n<1/2$, with z and a real numbers and $\epsilon\to 0$ is a small control parameter utilized to set the scale of the laminar phase. The laminar motion has its origin at points $x_n\sim N$ where $x_{n+1}\sim x_n\pm 1$ and thus the orbit is transfered to the same position in the neighboring cell. In a continuous time this corresponds to ballistic motion. Eventually the orbit leaves the region $x\sim N$ and the dynamics becomes chaotic. The duration of the chaotic phase is typically much shorter than the ballistic one. We mention that due to universality [15], intermittency appears for any $f(x)$ with a Taylor expansion for $x\ll 1$ given by the above relation. In Ref. [14] it was found that the density of probability of staying in the laminar phase a time t (or a distance r) has a power law tail

$$\psi(t,r) \sim \frac{\hat{b}}{(1+r)^{\mu}}\delta(|r|-\hat{b}t), \tag{1}$$

where $\hat{b}=2^z a+\epsilon/2$ and $\mu=z/(z-1)$. Indeed the motion is superdiffusive [14] for $2<\mu<3$ and ballistic for $1<\mu\leqslant 2$. Based on the above result, Zumofen and Klafter [19] calculated the probability $P(r,t)$ to be at location r at time t in the framework of the continuous random walk model

$$P(s,k) \sim \frac{1}{is+b|k|^{\mu-1}} 2\leqslant\mu\leqslant 3, \qquad \frac{s^{\mu-2}}{is^{\mu-1}+b|k|^{\mu-1}} 1<\mu<2, \tag{2}$$

where $P(s,k)$ is the Fourier transform of $P(x,t)$ and $b\sim\hat{b}$ up to factors of order the unity. For $2<\mu\leqslant 3$, the above equation is a solution of the following fractional Fokker Planck equation (for a recent review see Ref. [20]),

$$\frac{\partial P(x,t)}{\partial t} - \frac{b}{2}\frac{\partial^{\mu-1}}{\partial|x|^{\mu-1}}P(x,t) = \delta(x)\delta(t), \tag{3}$$

where we define the fractional derivative as in Ref. [20].

The phenomenon of intermittency in Hamiltonian systems has been related [21,22] to the presence of cantori [23] in phase space. A trajectory is typically trapped in the intricate cantori structures for times "long" as compared with those involved in the transport through the pure chaotic phase space. Meiss and co-workers [21] described the transport through cantori in phase space as a random walk in a Bethe lattice. Such a simplified model predicts a power law tail for the waiting times inside a cantori in agreement with numerical simulations [24]. Details of the distribution as the decay exponent are sensitive to the scaling properties of the cantori which depend on the considered energy and are thus nonuniversal. Since waiting times in phase space lead to ballistic motion in real space [22], a trajectory in real space is a mixture of long ballistic flights governed by a power-law distribution eventually interrupted by random walks when the trajectory escapes from the cantori region. The above simplified model cannot, in principle, describe the full complexity of typical KAM Hamiltonians with a mixed phase space where cantori with different scaling properties coexist with pure chaotic and integrable components (for a recent investigation in this direction see Ref. [25]). A different situation occurs in Hamiltonians in which, due to the nonanalyticity of the classical potential, the KAM theorem does not hold. In certain cases [26], as a parameter is switched on, the whole classical phase space undergoes an abrupt transition from integrable to homogeneously filled with cantori. The absence of additional classical structures permits in this case utilize the formalism above explained though the details of $P(r,t)$ may depend on the considered energy. Others class of systems with similar properties is that of pseudointegrable billiards [16], the phase space is also fractal and the classical motion presents anomalous transport properties [17,18].

II. QUANTUM MANIFESTATIONS OF CLASSICAL INTERMITTENCY

We now investigate quantum manifestations of classical intermittency by using semiclassical techniques. Our starting point is the study of the connected two level correlation function

$$R_2(s,g) = \Delta^2\langle\rho(\epsilon-s/2)\rho(\epsilon+s/2)\rangle - 1, \tag{4}$$

where $\rho(\epsilon)$ is the density of states at energy ϵ, Δ is the mean level spacing, the energy s is expressed in units of Δ, and the averaging is over an ensemble for disordered systems and over an interval of energy for single deterministic chaotic systems. The spectral properties depend on the ratio $g=E_c/\Delta$. In the ergodic limit $g\to\infty$, the Hamiltonian can be accurately approximated by a random matrix with the appropriate symmetry and WD statistics applies. For instance, for broken time reversal invariance $R_2(s,0)=-\sin^2(\pi s)/\pi^2 s^2$. Perturbative corrections ($g\gg 1$ and $s\gg 1$) to this result are evaluated [11] by simple perturbation theory,

$$R_2^{\text{pert}}(s,g) = \frac{1}{2\pi^2}\,\text{Re}\sum_n P^2(\epsilon_n,s),$$

where $P(\epsilon_n,s)$ is the propagator of the diffusion equation [in our case Eq. (2)], ϵ_n are the eigenvalues of the diffusion equation, and n runs over the integers. As expected, the lowest mode $n=0$ reproduces the asymptotic form of the WD statistics. Non perturbative corrections (leading to the oscillating terms) to this result have been worked out by using the supersymmetry method [4]. This approach is valid only in the $s\gg 1$ limit except for systems with broken time reversal invariance where it is supposed to hold for any s. In this case,

$$R_2(s,g) = R_2^{\text{pert}}(s,g) + R_2^{\text{osc}}(s,g)$$
$$R_2^{\text{osc}}(s,g) = \cos(2\pi s)\frac{D(s,g)}{2s^2\pi^2}, \tag{5}$$

where $D(s,g)$ is the spectral determinant associated to the diffusion equation. For normal diffusion $D^{-1}(s,g)=\Pi_{n\neq 0}(1+s^2/\epsilon_n^2)$ and $\epsilon_n=gn^2$. In principle one may ask whether the above formalism is applicable to the case of anomalous diffusion. It turns out that for $\mu \geqslant 2$ this can be demonstrated by mapping it onto an Anderson model with long range disorder [28] (see below). For $\mu<2$, although the mapping is in principle possible, we are not aware of a rigorous proof so our results should be considered in this case as a conjecture.

The spectral correlations associated with classical intermittency are now studied by using the above semiclassical techniques. We evaluate Eq. (5) for different μ and then we investigate the long range spectral correlations by the analysis of the number variance. We recall that the number variance $\Sigma^2(L)=\langle L\rangle^2-\langle L^2\rangle=L+2\int_0^L ds(L-s)R_2(s,g)$ measures the stiffness of the spectrum. In the metallic regime, for eigenvalues separated less than the Thouless energy, fluctuations are small and $\Sigma^2(L)\sim\log(L)$ for $L\gg 1$. Beyond the Thouless energy spectral fluctuations get stronger and $\Sigma^2(L)\sim L^{d/2}$ where d is the dimensionality of the space. For disorder strong enough eigenvalues are uncorrelated (Poisson statistics) and $\Sigma^2(L)=L$. At the MIT, the number variance is asymptotically proportional to χL ($\chi<1$) with χ being related with the multifractal scaling of the wave function moments [27].

A. Case I: $1<\mu<2$

This case corresponds with classical ballistic diffusion $\langle r^2\rangle\propto t^2$ [14]. The propagator of the diffusion equation is given by Eq. (2), $P(\epsilon_n,s)=s^{\mu-2}/(is^{\mu-1}+\epsilon_n)$, $D^{-1}(s,g)=\Pi_{n\neq 0}(1+s^{2\mu-2}/\epsilon_n^2)$, $\epsilon_n=g|n|^{\mu-1}$ (where periodic boundary condition and assumed) and, by using Eq. (5), $R_2(s,g)\sim g^{(\mu-1)^{-1}}/s$ for $s\gg 1$. The parameter b in Eq. (2) is related to g as follows. For normal diffusion $E_c\sim b/L^2$ and $g\sim bL^{d-2}$. However, in our case, since the diffusion is ballistic, $E_c\sim b^{2\mu-3}/L$ and $g\sim b^{2\mu-3}$ for $1<\mu<2$. We remark that the scale invariance of g may be modified by quantum corrections as in the case of a two-dimensional weekly disordered conductor. The number variance behaves asymptotically as $\Sigma^2(L)\sim g^{(\mu-1)^{-1}}L\log L$ with a subleading linear term as at the MIT. Additional work on the wavefunctions is needed to fully characterize the quantum properties in this region.

B. Case II: $\mu=2$

In this case, $D^{-1}(s,g)=\Pi_{n\neq 0}(1+s^2/n^2g^2)=(1/g^2)\times[\pi^2s^2/\sinh^2(\pi s/g)]$ and $R_2(s,g)$ can be explicitly evaluated,

$$R_2(s,h) = h^2\frac{\sin^2(\pi s)}{\sinh^2(\pi hs)}, \tag{6}$$

where $h=1/g\ll 1$. Remarkably, this correlation function has been put forward as a definition of critical statistics [9]. It reproduces typical features at the MIT as level repulsion (for $s\ll 1$ coincides with Wigner-Dyson statistics) and sub-Poisson number variance [$\Sigma^2(L)\sim hL$ for $L\gg 1$]. In this case $h\sim 1/b$ is also scale invariant. As discussed below, by mapping this case onto a Anderson model with long range disorder one can show that the wave functions are indeed multifractal as at the MIT. Finally, we remark that the classical motion associated to $\mu=2$ leads to $1/f$ noise [22].

C. Case III: $2<\mu\leqslant 3$

Now the dynamics is superdiffusive but sub-ballistics, $\langle r^2\rangle\sim t^{4-\mu}$ [24]. Classical anomalous diffusion is described by the propagator $P(\epsilon_n,s)=1/(is+\epsilon_n)$, $D^{-1}(s,g)=\Pi_{n\neq 0}(1+s^2/\epsilon_n^2)$ with $\epsilon_n=g|n|^{\mu-1}$. The asymptotic behavior of $R_2(s,g)\sim s^{-2+1/(\mu-1)}$ is power law instead of exponential. The conductance g is scale dependent and decreases as the system size increases, $E_c\sim L^{-2/(4-\mu)}$, $g\sim L^{1-2/(4-\mu)}$. For $s\gg g$, the power law tail of $R_2(s,g)$ leads to $\Sigma^2(L)\sim L^{1/(\mu-1)}$. Both the scaling of g and the spectral properties resemble those of a weakly disordered conductor in $d=2/(\mu-1)$ [28] dimensions. Finally for $\mu=3$ one recovers the expected behavior $\Sigma^2(L)\sim\chi\sqrt{L}$ of a 1D weakly disordered conductor (no anomalous diffusion).

We mention that similar findings have been reported in Anderson models with long range $1/r^{\mu-1}$ hopping [28]. For $2\leqslant\mu\leqslant 3$ the classical transport is also described by Eq. (2) provided that the disorder is weak enough [29]. By using the supersymmetry method not only the spectral correlations but also the wave functions can be studied analytically. It turns out that the eigenfunctions are power law localized [28] $|\psi(r)|\sim r^{-\mu+1}$. In the thermodynamic limit, they become localized for $\mu>2$, delocalized for $\mu<2$, and multifractal for $\mu=2$ as at the MIT. Finally, we point out that typical features of these long range hopping models as power law localization and criticality also appears in higher dimensions [28]. Thus, in d dimensions, wave functions are power-law localized for any exponent μ and multifractal for $\mu=d+1$.

We now discuss under what conditions the above findings are relevant for deterministic Hamiltonians. Obviously a first constraint is that the classical dynamics be described by Eq. (2). As discussed previously this could be the case for non-KAM systems [26] with a classical phase space homogeneously filled by cantori. Although the parameters defining Eq. (2) may depend on the considered energy, numerical results [22] suggest that the in certain cases they are barely modified in a broad window of energy and are close to the ones leading to $1/f$ noise. From a quantum mechanical point view our results are only valid in the limit $g\gg 1$ where interference (not tunneling) is the dominant quantum feature. In the context of quantum chaos this scale corresponds with $\hat{g}\sim\Delta W/\hbar$, where $\hbar$ is the Planck constant and ΔW is the flux swept across the cantorus with less flux at a given energy in one iteration of the map. Our results are thus applicable for energies such that $\hat{g}\gg 1$. This limit corresponds to the case when quantum mechanics can resolve the classical barrier. For $\hat{g}\sim 1$ quantum mechanics cannot resolve the gap and in order to cross it must tunnel through it. We mention that recently it has been reported [30] that the high energy excitations of non-KAM systems such as the anisotropic Kepler problem or the Coulomb billiard [26] are correlated according to critical statistics. It would be interesting to check whether the classical mechanics of these systems present $1/f$ noise as predicted in this paper.

In conclusion, we have investigated quantum manifestation of classical intermittency. It has been shown that for classical ballistic diffusion and $1/f$ noise (a special case of intermittency) the spectral correlations are given by critical statistics as at the MIT. In other cases the classical motion is superdiffusive but sub-ballistic, the wave functions are power law localized and the spectral correlations are similar to those of a weakly disordered conductor in less than two dimensions. In the context of Hamiltonian systems we have suggested that these results may be relevant for the description of the spectral correlations of non-KAM systems with classical phase space homogeneously filled by cantori.

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